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A discrete integrable hierarchy related to the supersymmetric eigenvalue model

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Abstract. The supersymmetric eigenvalue model has been proposed as the analogue for two-dimensional supergravity of the matrix model. The double-scaling limit exhibits an integrable fermionic hierarchy which is related to the Korteweg–de Vries hierarchy. In this paper, a discrete analogue of this integrable structure, related to the Toda lattice, is found in the supersymmetric eigenvalue model.

1. Introduction

Recently, important progress has been made in the understanding of non-critical string theories, requiring, as it does, the analysis of the quantum theory of two-dimensional gravity coupled to conformal-matter systems. Insights have been achieved by a variety of methods, one of the most significant being a discrete approach to two-dimensional quantum gravity in the form of matrix models where the double-scaling limit allows extraction of information about the continuum theory. The analogue of the matrix-model approach for quantum supergravity coupled to superconformal-matter systems is not obvious, due to the technical problem of defining a discrete analogue of the gravitino. A proposal by Alvarez-Gaumé *et al* [1] to avoid this problem characterizes the partition function of the discrete version of supergravity in terms of a set of super-Virasoro constraints, analogous to the Virasoro constraints satisfied by the partition function of the one-matrix model. A solution to these constraints was found in the form of a supersymmetric eigenvalue model, which is a supersymmetric generalization of the eigenvalue model which results when the 'angular' integrations are carried out in the ordinary one-matrix model. The double-scaling limit of the supersymmetric eigenvalue model reproduces the scaling dimensions of gravitationally dressed operators for the $(2, 4m)$ minimal superconformal models coupled to supergravity, an indication that this approach may be correct.

One of the more surprising results from matrix models has been the revelation of links between quantum two-dimensional gravity and integrable systems, specifically the Korteweg–de Vries (KdV) hierarchy. The integrable structure is already present at the level of the matrix model where the partition function is a tau function of the Toda hierarchy satisfying a set of Virasoro constraints. An important feature of the double-scaling limit, which relates the one-matrix model to two-dimensional gravity, is that this integrable structure is preserved.

In the case of the double-scaling limit of the supersymmetric eigenvalue model, an underlying integrable structure was identified by Becker and Becker [2] and examined further in [3] and [4]. It is an integrable fermionic hierarchy which essentially results from variation of the KdV hierarchy. In order to better understand the relationship between the supersymmetric eigenvalue model and two-dimensional supergravity, it is important to know if this integrable structure observed in the continuum limit of the supersymmetric eigenvalue model has some progenitor in the discrete theory which is preserved in the process of taking the double-scaling limit. In this paper, it is shown that this is indeed the case, in that an integrable fermionic extension of the Toda hierarchy can be related to the partition function of the supersymmetric eigenvalue model.

The paper is organized in the following fashion. In section 2, the one-matrix model and its relation to the Toda hierarchy is briefly reviewed, while in section 3 some facts about the supersymmetric eigenvalue model and the integrable structure underlying its double-scaling limit are presented. In the next section, an integrable fermionic extension of the Toda hierarchy is derived and its relation to the partition function of the supersymmetric eigenvalue model is examined in section 5. The conclusion contains some observations on the nature of the super-Virasoro constraints on the partition function of the supersymmetric eigenvalue model in the light of this integrable structure.

2. The bosonic one-matrix model

The bosonic one-matrix model and its relationship to two-dimensional-matter systems coupled to quantum gravity has been much studied—we refer to the reviews [5] and the references contained therein for details. The partition function involves a matrix integral over $N \times N$ Hermitian matrices and, after integrating over the 'angular variables', takes the form

$$Z_{\mathrm{B}}[g_k; N] = \left(\prod_{i=1}^{N} \int \mathrm{d}\lambda_i \, \mathrm{e}^{-V(\lambda_i)}\right) \Delta(\lambda)^2. \tag{1}$$

The surviving integrals are over the eigenvalues λ_i $(i = 1, 2, \ldots, N)$ of the Hermitian matrices, while the potential is

$$V(\lambda) = \sum_{k0}^{\infty} g_k \lambda^k$$

and

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

is the Vandermonde determinant. The partition function can be re-expressed in the compact form

$$Z_{\mathrm{B}}[g_k; N] = N! \det_{i,j} H_{i,j} \tag{2}$$

where $H_{i,j}$ is the $N \times N$ matrix with entries $H_{i,j} = H_{i+j-2}$ for

$$H_i = \int \mathrm{d}\lambda \, \mathrm{e}^{-V(\lambda)} \lambda^i \qquad (i \geqslant 0). \tag{3}$$

The partition function $Z_B[g_k; N]$ has some remarkable properties. First, it satisfies a set of Virasoro constraints $L_n Z_B[g_k; N] = 0$, $n \geqslant -1$ [6–9] with the Virasoro generators expressed in terms of the coupling constants g_k of the matrix potential by

$$L_n = \sum_{k=0}^{n} \frac{\partial^2}{\partial g_{n-k} \partial g_k} + \sum_{k \geqslant 0} k g_k \frac{\partial}{\partial g_{k+n}}. \tag{4}$$

It is also a tau function for the Toda-lattice hierarchy [8, 10, 11], a discrete integrable system.

The relationship with the Toda hierarchy is conventionally established using the (unnormalized) orthogonal polynomials $p_n(\lambda)$ defined by $p_n(\lambda) = \lambda^n + O(\lambda^{n-1})$ and

$$\int d\lambda \, e^{-V(\lambda)} p_m(\lambda) p_n(\lambda) \delta_{m,n} = h_n. \tag{5}$$

The orthogonal polynomials satisfy a recursion relation

$$\lambda p_n(\lambda) = p_{n+1}(\lambda) + S_n p_n(\lambda) + R_n p_{n-1}(\lambda) \tag{6}$$

where the R_n and S_n are independent of λ but depend on the couplings g_k via the coupling constant dependence of the orthogonal polynomials. The latter can be expressed in the form

$$\frac{\partial p_n(\lambda)}{\partial g_k} = \sum_{m0}^{n-1} = \gamma_{nm}^{(k)} p_m(\lambda) \tag{7}$$

with the coefficients $\gamma_{nm}^{(k)}$ being determined in terms of the R_i and S_i by differentiation of (5) and repeated use of (6). Equations (6) and (7) form a Lax pair for an integrable hierarchy in that their compatibility (in the sense that $[\partial/\partial g_k, \lambda] = 0$) gives rise to a sequence of flow equations for R_n and S_n. The first few of these are

$$\begin{aligned}
\frac{\partial R_n}{\partial g_1} &= R_n(S_{n-1} - S_n) \\
\frac{\partial S_n}{\partial g_1} &= R_n - R_{n+1} \\
\frac{\partial R_n}{\partial g_2} &= R_n(R_{n-1} - R_{n+1} + S_{n-1}^2 - S_n^2) \\
\frac{\partial S_n}{\partial g_2} &= R_n(S_n + S_{n-1}) - R_{n+1}(S_{n+1} + S_n).
\end{aligned} \tag{8}$$

This hierarchy of equations is the Toda-lattice hierarchy and it can also be expressed in Hamiltonian form. Namely, there exist Poisson brackets $\{\ ,\ \}_{PB}$ and Hamiltonians $\mathcal{H}_m$ such that the flow equations can be written as

$$\frac{\partial R_n}{\partial g_m} = \{\mathcal{H}_{m+1}, R_n\}_{PB} \qquad \frac{\partial S_n}{\partial g_m} = \{\mathcal{H}_{m+1}, S_n\}_{PB}. \tag{9}$$

The Poisson brackets are defined by

$$\begin{aligned}
\{R_m, R_n\}_{PB} &= 0 \\
\{R_m, S_n\}_{PB} &= R_m(\delta_{m,n} - \delta_{m,n+1}) \\
\{S_m, S_n\}_{PB} &= 0
\end{aligned} \tag{10}$$

and the Hamiltonians $\mathcal{H}_m$ are of the form $\mathcal{H}_m = \sum_n \mathcal{H}_m(n)$ with

$$\mathcal{H}_m(n) = \frac{1}{m h_n} \int d\lambda \, e^{-V(\lambda)} p_n(\lambda) \lambda^m p_n(\lambda).$$

The first few are

$$\begin{aligned}
\mathcal{H}_1 &= \sum_n S_n \\
\mathcal{H}_2 &= \tfrac{1}{2} \sum_n (R_{n+1} + S_n^2 + R_n) \\
\mathcal{H}_3 &= \tfrac{1}{3} \sum_n (R_{n+1} S_{n+1} + 2R_{n+1} S_n + S_n^3 + 2R_n S_n + R_n S_{n-1}).
\end{aligned} \tag{11}$$

The flows in parameters g_m and g_n are compatible (i.e. they commute) because the Hamiltonians are in involution with respect to the Poisson brackets, $\{\mathcal{H}_{m+1}, \mathcal{H}_{n+1}\}_{\mathrm{PB}} = 0$.

The existence of the infinite hierarchy of compatible flows (or equivalently, an infinite set of conserved charges $\mathcal{H}_m$ in involution) is related to the existence of a bi-Hamiltonian structure. This means that there is a second coordinated Poisson structure $\{\ ,\ \}_{\mathrm{PB2}}$ such that the hierarchy can be expressed in the form

$$\frac{\partial R_n}{\partial g_m} = \{\mathcal{H}_m, R_n\}_{\mathrm{PB2}} \qquad \frac{\partial S_n}{\partial g_m} = \{\mathcal{H}_m, S_n\}_{\mathrm{PB2}}.$$

The second Poisson structure is defined by

$$\begin{aligned}
\{R_m, R_n\}_{\mathrm{PB2}} &= R_m R_n (\delta_{m,n-1} - \delta_{m,n+1}) \\
\{R_m, S_n\}_{\mathrm{PB2}} &= R_m S_n (\delta_{m,n} - \delta_{m,n+1}) \\
\{S_m, S_n\}_{\mathrm{PB2}} &= R_n \delta_{m,n-1} - R_m \delta_{m,n+1}.
\end{aligned} \tag{12}$$

The relationship between the one-matrix model and the Toda hierarchy can be expressed at a more fundamental level—the partition function $Z_{\mathrm{B}}[g_k; N]$ is a tau function $\tau_N[g_k]$ for the hierarchy. The corresponding Baker functions are essentially the orthogonal polynomials [8, 10], in that

$$p_n(\lambda) = \lambda^n \frac{X(\lambda) \cdot \tau_n[g_k]}{\tau_n[g_k]} \tag{13}$$

with $X(\lambda)$ the vertex operator

$$X(\lambda) = \exp\left(\sum_{k>0} \frac{1}{k} \lambda^{-k} \frac{\partial}{\partial g_k} \right).$$

The solutions R_n and S_n to the hierarchy of equations (9) generated by the matrix model via the recursion relation (6) take the form

$$\begin{aligned}
R_n &= \frac{\partial^2}{\partial g_1^2} \ln \tau_n[g_k] \\
S_n &= -\frac{\partial}{\partial g_1} \ln \frac{\tau_{n+1}[g_k]}{\tau_n[g_k]}.
\end{aligned} \tag{14}$$

Using expression (2) for the partition function and the fact that, from the definition (3),

$$\frac{\partial H_i}{\partial g_k} = -H_{i+k}$$

it is clear that the R_n and S_n can be expressed entirely in terms of the H_i. Further, the fact that that the polynomial in λ, given by (13), is of maximum order λ^n and contains no negative powers of λ is a consequence of (2) and

$$X(\lambda) \cdot H_i = H_i - \frac{H_{i+1}}{\lambda}.$$

3. The supersymmetric eigenvalue model

Although a direct analogue of the matrix-model formulation for theories of two-dimensional superconformal matter coupled to quantum supergravity is lacking at present due to the conceptual difficulty of interpreting the gravitino in discrete terms, Alvarez-Gaumé *et al* [1] have found evidence that a supersymmetric analogue of the 'eigenvalue model' (1) may be relevant to two-dimensional supergravity theories. The model is defined by introducing *fermionic partners θ_i and $\xi_{k+\frac{1}{2}}$ for the eigenvalues λ_i and coupling constants g_k, respectively,* and requiring that the partition function $Z_S[g_k, \xi_{k+\frac{1}{2}}; N]$ satisfy the set of super-Virasoro constraints

$$G_{n-\frac{1}{2}} Z_S[g_k, \xi_{k+\frac{1}{2}}; N] = 0 \qquad n \geqslant 0$$

with

$$G_{n-\frac{1}{2}} = \sum_{k=0}^{\infty} \xi_{k+\frac{1}{2}} \frac{\partial}{\partial g_{k+n}} + \sum_{k=0}^{\infty} k g_k \frac{\partial}{\partial \xi_{k+n-\frac{1}{2}}} + \sum_{k=0}^{n-1} \frac{\partial}{\partial \xi_{k+\frac{1}{2}}} \frac{\partial}{\partial g_{n-k-1}}. \tag{15}$$

A solution to this set of constraints is given by

$$Z_S[g_k, \xi_{k+\frac{1}{2}}; N] = \left(\prod_{i=1}^{N} \int \mathrm{d}\lambda_i \int \mathrm{d}\theta_i \, \mathrm{e}^{-V(\lambda_i, \theta_i)} \right) \prod_{i<j} (\lambda_i - \lambda_j - \theta_i \theta_j) \tag{16}$$

where the supersymmetric potential is

$$V(\lambda, \theta) = \sum_{k=0}^{\infty} (g_k \lambda^k + \xi_{k+\frac{1}{2}} \theta \lambda^k). \tag{17}$$

The partition function is non-vanishing only for even N and it was shown in [12] that

$$Z_S[g_k, \xi_{k+\frac{1}{2}}; 2N] = (-1)^N (2N)! \, \mathrm{Pfaff} \, A \exp\left(-\tfrac{1}{2} \sum_{i,j=1}^{2N} \zeta_i (A^{-1})_{i,j} \zeta_j \right) \tag{18}$$

where

$$\zeta_i = \sum_{k \geqslant 0} \xi_{k+\frac{1}{2}} H_{i+k-1} \qquad (i \geqslant 1) \tag{19}$$

and A is the $2N \times 2N$ antisymmetric matrix with entries

$$A_{i,j} = -\tfrac{1}{2} \sum_{k0}^{j-i-1} H_{i+k-1} H_{j-k-2} \qquad (i < j).$$

This expression for the partition function is the analogue of (2) in the bosonic case as the ζ_i are natural fermionic analogues of the H_i, in that

$$\int \mathrm{d}\lambda\, \mathrm{e}^{-V(\lambda,\theta)} \lambda^i = H_i + \theta \zeta_{i+1}. \tag{20}$$

An equivalent expression for the partition function is

$$Z_{\mathrm{S}}[g_k, \xi_{k+\frac{1}{2}}; 2N] = Z_{\mathrm{B}}[g_k; N]^2 \exp\left(-2 \sum_{k,n\geqslant 0} \xi_{k+\frac{1}{2}} \xi_{m+\frac{1}{2}} \frac{\partial^2 \ln Z_{\mathrm{B}}[g_k; N]}{\partial g_{k+1} \partial g_m} \right) \tag{21}$$

which was obtained by Becker and Becker [2] by evaluating (16) to second order in the fermionic couplings and relates the partition function of the supersymmetric $2N$-eigenvalue model to that of the bosonic N-eigenvalue model.

On the basis of result (21), Becker and Becker [2] have found a solution to the super-Virasoro constraints in the double-scaling limit [1] in terms of the double-scaling limit $Z_{\mathrm{B}}[t_k]$ of the partition function of the one-matrix model, namely

$$Z_{\mathrm{S}}[t_k, \hat{t}_k] = Z_{\mathrm{B}}[t_k] \exp\left(-\sum_{n\geqslant 0} \sum_{m\geqslant=1} \hat{t}_n \hat{t}_k \frac{\partial^2 \ln Z_{\mathrm{B}}[t_k]}{\partial t_n \partial t_{m-1}} \right)$$

where t_k and $\hat{t}_k$ are the continuum bosonic and fermionic couplings, respectively. As is now well appreciated [5], $Z_{\mathrm{B}}[t_k]$ is the square† of a tau function $\tau[t_k]$ for the KdV hierarchy of nonlinear differential equations. The integrable flow equations of the KdV hierarchy are expressed in terms of the variable $u D^2 \ln Z_{\mathrm{B}}[t_k]$ (with $D = \partial/\partial x$, where $x \equiv t_0$) and take the form

$$\frac{\partial u}{\partial t_n} = D\mathcal{R}_{n+1} \tag{22}$$

where the $\mathcal{R}_{n+1} = (\partial/\partial t_n) D \ln Z_{\mathrm{B}}[t_k]$ are the Gelfand–Dikii polynomials (in u and its derivatives) generated by the recursion relation

$$D\mathcal{R}_{n+1} = (D^3 + 4uD + 2(Du))\mathcal{R}_n$$

with $\mathcal{R}_0 - \frac{1}{2}$. Using these recursion relations, the free energy $F_{\mathrm{S}}[t_k, \hat{t}_k] = \ln Z_{\mathrm{S}}[t_k, \hat{t}_k]$ of the supersymmetric eigenvalue model can be expressed in terms of the fermionic variable

$$\hat{t} = -\sum_{k\geqslant 0} \hat{t}_k \mathcal{R}_k \tag{23}$$

as

$$D^2 F_{\mathrm{S}}[t_k, \hat{t}_k] = u - 2D(\hat{t} D\hat{t}). \tag{24}$$

† In the discrete and continuum cases, the part of the supersymmetric partition function that is independent of the fermionic couplings is the *square* of a tau function for an integrable hierarchy.

In [2], it was shown that u and $\hat{t}$ satisfy a hierarchy of flow equations in the even and odd couplings t_k and $\hat{t}_k$ in which the flows are expressed as polynomials in u and $\hat{t}$, their integrability being established by the existence of specific solutions generated by $Z_S[t_k, \hat{t}_k]$. If the double-scaling limit of the supersymmetric eigenvalue model does describe quantum supergravity coupled to supersymmetric matter, then these equations are related to Ward identities satisfied by correlation functions. The flow equations for u and $D\hat{t}$ were shown in [3] to constitute a variation of the KdV hierarchy and, based on this work, it was established in [4] that there exists a bi-Hamiltonian formulation in terms of odd Poisson brackets similar to the antibracket of Batalin and Vilkovisky [13]. This is intimately related to the integrability.

To further the understanding of the relationship between the supersymmetric eigenvalue model and the double-scaling limit proposed in [2], it is important to be able to identify the discrete analogue of this integrable structure.

4. The discrete integrable fermionic hierarchy

For the proposed double-scaling limit of the supersymmetric eigenvalue model discussed above, Figueroa and O'Farrill [3] noted that the hierarchy of equations describing the bosonic flows for u and $D\hat{t}$ in the variable t_n follow by replacing u in the KdV flows

$$\frac{\partial u}{\partial t_n} = D\mathcal{R}_{n+1}$$

by

$$U = u + \theta D\hat{t} \tag{25}$$

(where θ is a Grassmann parameter) and identifying coefficients of 1 and θ on both sides of the equation. It was demonstrated in [4] that the resulting equations can be written in Hamiltonian form with respect to *odd* Poisson brackets, analogous to the antibracket in the formalism of Batalin and Vilkovisky [13]. This is an infinite-dimensional analogue of a natural Hamiltonian dynamics which is induced on the cotangent bundle to a finite-dimensional symplectic manifold by Hamiltonian dynamics on the manifold [14]. Roughly speaking, it is the fact that the flows for $D\hat{t}$ are, by the construction (25), the 'first variations' of the usual KdV flows which makes this interpretation appropriate.

This suggests looking at the first variation of the Toda hierarchy to identify the discrete fermionic hierarchy which is related to the supersymmetric eigenvalue model. This is achieved by replacing R_n and S_n on the left- and right-hand sides of the Toda hierarchy of equations (8) by $R_n + \theta\hat{R}_n$ and $S_n + \theta\hat{S}_n$, where hatted quantities are Grassmann-odd, and equating the coefficients of 1 and θ on each side of the equations. The first few equations for $\hat{R}_n$ and $\hat{S}_n$ are

$$\begin{aligned}
\frac{\partial\hat{R}_n}{\partial g_1} &= \hat{R}_n(S_{n-1} - S_n) + R_n(\hat{S}_{n-1} - \hat{S}_n)\\
\frac{\partial\hat{S}_n}{\partial g_1} &= \hat{R}_n - \hat{R}_{n+1}\\
\frac{\partial\hat{R}_n}{\partial g_2} &= \hat{R}_n(R_{n-1} - R_{n+1} + S_{n-1}^2 - S_n^2) + R_n(\hat{R}_{n-1} - \hat{R}_{n+1} + 2S_{n-1}\hat{S}_{n-1} - 2S_n\hat{S}_n)\\
\frac{\partial\hat{S}_n}{\partial g_2} &= \hat{R}_n(S_n + S_{n-1}) - \hat{R}_{n+1}(S_{n+1} + S_n) + R_n(\hat{S}_n + \hat{S}_{n-1}) - R_{n+1}(\hat{S}_{n+1} + \hat{S}_n).
\end{aligned} \tag{26}$$

This hierarchy of equations admits a Hamiltonian structure with respect to odd Poisson brackets.

Before showing this, we briefly review the construction of odd Poisson brackets on finite-dimensional symplectic manifolds [14]. Given an even-dimensional manifold M with local coordinates x^i and a non-degenerate closed two-form $\omega = \omega_{ij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j$, ordinary Poisson brackets are defined by $\{f, g\}_{\mathrm{PB}} = (\partial f/\partial x^i)\omega^{ij}(x)\partial g/\partial x^j$ where f and g are functions on M and $\omega^{ij}\omega_{jk} = \delta^i_k$. A flow in the parameter t on the space of functions on M is Hamiltonian if there is a function $\mathcal{H}(x)$ such that $\partial f/\partial t = \{\mathcal{H}, f\}_{\mathrm{PB}}$. An odd symplectic structure can be defined on the tangent bundle of M with local coordinates (x^i, ξ_i) (where the Grassmann variable ξ_i corresponds to $\partial/\partial x^i$) and gives rise to the odd Poisson brackets

$$\{x^i, x^j\}_{\mathrm{AB}} = 0 = \{\xi_i, \xi_j\}_{\mathrm{AB}} \qquad \{x^i, \xi_j\}_{\mathrm{AB}} = -\delta^i_j$$

(with 'AB' standing for antibracket). In terms of $\xi^i = \omega^{ij}(x)\xi_j$, this becomes

$$\{x^i, x^j\}_{\mathrm{AB}} = 0 \qquad \{x^i, \xi^j\}_{\mathrm{AB}} = \omega^{ij} = \{\xi^i, x^j\}_{\mathrm{AB}} \qquad \{\xi^i, \xi^j\}_{\mathrm{AB}} = \sum_k \frac{\partial \omega^{ij}}{\partial x^k}\xi^k. \tag{27}$$

The ordinary Hamiltonian flow

$$\frac{\partial x^i}{\partial t} = \{\mathcal{H}, x^i\}_{\mathrm{PB}} = \sum_j \frac{\partial \mathcal{H}(x)}{\partial x^j}\omega^{ji}(x) \tag{28}$$

can be naturally extended to the cotangent bundle of M (with local coordinates (x^i, ξ^i)) by replacing x^i in (28) by $X^i = x^i + \theta\xi^i$ and taking the coefficients of $\mathbf{1}$ and θ

$$\frac{\partial x^i}{\partial t} = \sum_j \frac{\partial \mathcal{H}}{\partial x^j}\omega^{ji}$$

$$\frac{\partial \xi^i}{\partial t} = \sum_{j,k}\left(\frac{\partial \mathcal{H}}{\partial x^j}\frac{\partial \omega^{ji}}{\partial x^l} + \frac{\partial^2 \mathcal{H}}{\partial x^j \partial x^l}\omega^{ji}\right)\xi^l.$$

By defining $\hat{\mathcal{H}}(x)$ to be the coefficient of θ in $\mathcal{H}(X)$, this system of equations can be put into Hamiltonian form with respect to the odd Poisson bracket as [14]

$$\frac{\partial x^i}{\partial t} = \{\hat{\mathcal{H}}, x^i\}_{\mathrm{AB}} \qquad \frac{\partial \xi^i}{\partial t} = \{\hat{\mathcal{H}}, \xi^i\}_{\mathrm{AB}}.$$

The hierarchy of equations obtained by first variation of the Toda hierarchy provides a (countably) infinite-dimensional example of this construction. If R_n and S_n in the Hamiltonians $\mathcal{H}_m$ for the Toda hierarchy are replaced by $R_n + \theta\hat{R}_n$ and $S_n + \theta\hat{S}_n$, respectively, and the coefficient of θ in the resulting expression is denoted by $\hat{\mathcal{H}}_m$, then the hierarchy of equations whose first few members are given by (8) and (26) can be expressed in the with respect to the odd Poisson brackets

$$\begin{aligned}
&\{R_m, R_n\}_{\mathrm{AB}} = 0 = \{R_m, S_n\}_{\mathrm{AB}} = \{S_m, S_n\}_{\mathrm{AB}} \\
&\{R_m, \hat{R}_n\}_{\mathrm{AB}} = 0 = \{S_m, \hat{S}_n\}_{\mathrm{AB}} \\
&\{R_m, \hat{S}_n\}_{\mathrm{AB}} = R_m(\delta_{m,n} - \delta_{m,n+1}) = \{\hat{R}_m, S_n\}_{\mathrm{AB}} \\
&\{\hat{R}_m, \hat{R}_n\}_{\mathrm{AB}} = 0 = \{\hat{S}_m, \hat{S}_n\}_{\mathrm{AB}} \\
&\{\hat{R}_m, \hat{S}_n\}_{\mathrm{AB}} = \hat{R}_m(\delta_{m,n} - \delta_{m,n+1})
\end{aligned} \tag{30}$$

which are associated with the Poisson brackets (10) by an infinite-dimensional version of the construction (27). This is verified explicitly in appendix A as it is not *a priori* obvious that the finite-dimensional results of [14] are applicable due to the infinite-dimensional nature of the symplectic manifold underlying the Toda hierarchy. It is also shown in appendix A that the flows in different parameters are compatible (in the sense that $[\partial/\partial g_m, \partial/\partial g_n] = 0$) as a result of the Jacobi identity for odd Poisson brackets and the fact that $\{\hat{\mathcal{H}}_{m+1}, \hat{\mathcal{H}}_{n+1}\}_{\mathrm{AB}} = 0$.

In fact, it is possible to define a second set of odd Poisson brackets $\{\ ,\ \}_{\mathrm{AB2}}$ based on the second Poisson structure (12) of the Toda hierarchy:

$$\begin{aligned}
&\{R_m, R_n\}_{\mathrm{AB2}} = 0 = \{R_m, S_n\}_{\mathrm{AB2}} = \{S_m, S_n\}_{\mathrm{AB2}} \\
&\{R_m, \hat{R}_n\}_{\mathrm{AB2}} = R_m R_n(\delta_{m,n-1} - \delta_{m,n+1}) \\
&\{R_m, \hat{S}_n\}_{\mathrm{AB2}} = R_m S_n(\delta_{m,n} - \delta_{m,n+1}) = \{\hat{R}_m, S_n\}_{\mathrm{AB2}} \\
&\{S_m, \hat{S}_n\}_{\mathrm{AB2}} = R_{m+1}\delta_{m,n-1} - R_m\delta_{m,n+1} \\
&\{\hat{R}_m, \hat{R}_n\}_{\mathrm{AB2}} = (R_m\hat{R}_n + \hat{R}_m R_n)(\delta_{m,n-1} - \delta_{m,n+1}) \\
&\{\hat{R}_m, \hat{S}_n\}_{\mathrm{AB2}} = (R_m\hat{S}_n + \hat{R}_m S_n)(\delta_{m,n} - \delta_{m,n+1}) \\
&\{\hat{S}_m, \hat{S}_n\}_{\mathrm{AB2}} = \hat{R}_{m+1}\delta_{m,n-1} - \hat{R}_m\delta_{m,n+1}.
\end{aligned} \tag{31}$$

In terms of this second odd Poisson structure, the equations (26) can be expressed in the form $\partial A_n/\partial g_m = \{\hat{\mathcal{H}}_m, A_n\}_{\mathrm{AB2}}$, where A_n represents R_n, S_n, $\hat{S}_n$ or $\hat{R}_n$. The verification is similar to that for the first odd Poisson structure. The existence of a bi-Hamiltonian structure for the hierarchy (26) can presumably be used to prove the existence of the infinitely many conserved quantities $\hat{\mathcal{H}}_m$ purely in terms of the odd Poisson structure rather than resorting to the use of the ordinary Poisson structure for the Toda hierarchy (which is a sub-hierarchy of (26)) as is done in the appendix.

In the next section, we find specific solutions to this hierarchy of equations related to the partition function of the supersymmetric eigenvalue model. The bi-Hamiltonian structure discussed above, which underlies this hierarchy of equations, is thus the discrete progenitor of the bi-Hamiltonian structure constructed in [3, 4] for the integrable hierarchy of Becker and Becker [2] which appears in the continuum limit of the supersymmetric eigenvalue model.

5. Relating the hierarchy to the supersymmetric eigenvalue model

First, we recall that the bosonic eigenvalue model generated solutions R_n and S_n to the Toda hierarchy (9) of the form (14) and that these solutions can be expressed entirely in terms of the H_i defined in (3). Using the earlier observation (20) that the ζ_{i+1} in (19) is the natural fermionic analogue of the H_i, we define $\hat{R}_n$ and $\hat{S}_n$ to be the coefficients of θ when the replacement $H_i \to H_i + \theta\zeta_{i+1}$ is made in these solutions:

$$R_n[H_i + \theta\zeta_{i+1}] = R_n[H_i] + \theta\hat{R}_n[H_i, \zeta_{i+1}].$$

Then R_n, S_n, $\hat{R}_n$ and $\hat{S}_n$, defined in this fashion, generate solutions to the hierarchy of equations (29). This follows because ζ_{i+1} can be written in the form δH_i, with $\delta = -\sum_{k\geqslant 0}\xi_{k+\frac{1}{2}}\partial/\partial g_k$, so that $\hat{R}_n = \delta R_n$. The absence of any g_k dependence in the operator δ means that $[\delta, \partial/\partial g_k] = 0$ with the result that

$$\frac{\partial\hat{R}_n}{\partial g_k} = \delta\frac{\partial R_n}{\partial g_k}.$$

Since $\partial R_n/\partial g_k$ can be expressed in terms of a polynomial in the R_n and the S_n, $\delta(\partial R_n/\partial g_k)$ is equivalent to the coefficient of θ after the replacements $R_n \to R_n + \theta\hat{R}_n$ and $S_n \to S_n + \theta\hat{S}_n$ are made in this polynomial, which is precisely the means by which the right-hand sides of the hierarchy of equations (26) were obtained (and similarly for $\partial\hat{S}_n/\partial g_k$).

It is also interesting to note that the solutions R_n, S_n, $\hat{R}_n$ and $\hat{S}_n$ (defined above) obey 'odd'-flow equations with respect to the fermionic couplings $\xi_{k+\frac{1}{2}}$. Trivially,

$$\frac{\partial R_n}{\partial\xi_{m+\frac{1}{2}}} = 0 = \frac{\partial S_n}{\partial\xi_{m+\frac{1}{2}}}. \tag{32}$$

Also, using $\hat{R}_n = \delta R_n$ and $\hat{S}_n = \delta S_n$ with $\delta = -\sum_{k\geqslant 0}\xi_{k+\frac{1}{2}}\partial/\partial g_k$

$$\frac{\partial\hat{R}_n}{\partial\xi_{m+\frac{1}{2}}} = -\frac{\partial R_n}{\partial g_m} \qquad \frac{\partial\hat{S}_n}{\partial\xi_{m+\frac{1}{2}}} = -\frac{\partial S_n}{\partial g_m}. \tag{33}$$

These flows can also be put into bi-Hamiltonian form with respect to the odd Poisson brackets as

$$\begin{aligned} -\frac{\partial R_n}{\partial\xi_{m+\frac{1}{2}}} &= \{\mathcal{H}_{m+1}, R_n\}_{\mathrm{AB}} = \{\mathcal{H}_m, R_n\}_{\mathrm{AB2}} \\ -\frac{\partial S_n}{\partial\xi_{m+\frac{1}{2}}} &= \{\mathcal{H}_{m+1}, S_n\}_{\mathrm{AB}} = \{\mathcal{H}_m, S_n\}_{\mathrm{AB2}} \\ -\frac{\partial\hat{R}_n}{\partial\xi_{m+\frac{1}{2}}} &= \{\mathcal{H}_{m+1}, \hat{R}_n\}_{\mathrm{AB}} = \{\mathcal{H}_m, \hat{R}_n\}_{\mathrm{AB2}} \\ -\frac{\partial\hat{S}_n}{\partial\xi_{m+\frac{1}{2}}} &= \{\mathcal{H}_{m+1}, \hat{S}_n\}_{\mathrm{AB}} = \{\mathcal{H}_m, \hat{S}_n\}_{\mathrm{AB2}}. \end{aligned} \tag{34}$$

This is proven in appendix A, along with the fact that these flows *anticommute* (since the variables $\xi_{k+\frac{1}{2}}$ are Grassmann-odd) with each other and *commute* with the even flows (29). Note that the Hamiltonians determining the odd flows are the Grassmann-even Hamiltonians $\mathcal{H}_m$ (as opposed to the Grassmann-odd $\hat{\mathcal{H}}_m$ for the even flows). It should be emphasized that the results in appendix A constitute a proof of the integrability of the fermionic hierarchy

(29), (34), which includes both even and odd flows, independent of the existence of the specific solutions given above.

To relate these solutions of the variation of the Toda hierarchy to the partition function of the supersymmetric eigenvalue model, recall that the partition function for the double-scaling limit proposed in [2] satisfies

$$D^2 \ln Z_S[t_n, \hat{t}_n] = u - 2D(\hat{t}D\hat{t})$$

with $D = \partial/\partial x \equiv \partial/\partial t_0$ and it is u and $D\hat{t}$ which constitute the solutions to the integrable 'first variation' of the KdV hierarchy. As the discrete analogue of $Df(x)$ is $f_n - f_{n-1}$ (this follows from the $N \times N$ Hermitian-matrix model underlying the supersymmetric eigenvlaue model [8, 10, 11]), so $D^2 \ln Z_S[t_n, \hat{t}_n]$ has the discrete analogue

$$\ln Z_S[g_k, \xi_{k+\frac{1}{2}}; 2n+2] - 2\ln Z_S[g_k, \xi_{k+\frac{1}{2}}; 2n] + \ln Z_S[g_k, \xi_{k+\frac{1}{2}}; 2n-2]. \tag{35}$$

In appendix B, this is shown to be equivalent to

$$2\ln R_n - 2\left(\frac{\partial \chi_n}{\partial g_1}\chi_n - \frac{\partial \chi_{n-1}}{\partial g_1}\chi_{n-1}\right) \tag{36}$$

with

$$\chi_n = \sum_{m\geqslant 0} \xi_{m+\frac{1}{2}} \frac{\partial}{\partial g_m} \ln \frac{\tau_{n+1}[g_k]}{\tau_n[g_k]}. \tag{37}$$

Comparing with the continuum case, $2\ln R_n$ is the discrete analogue of $u(x)$ (this is known from the bosonic case [8, 11]), χ_n is the discrete analogue of $\hat{t}(x)$ and $\partial\chi_n/\partial g_1$ is the discrete analogue of $D\hat{t}(x)$. Just as $D\hat{t}(x)$ is a solution to the 'first variation' of the KdV hierarchy in the continuum case, so $\partial\chi_n/\partial g_1$ is a solution to the first variation of the Toda hierarchy, as $\partial\chi_n/\partial g_1 = \hat{S}_n$. This follows from $Z_B[g_k; n] = \tau_n$ and (14), which imply that $\partial\chi_n/\partial g_1 = \delta S_n = \hat{S}_n$.

These correspondences between discrete and continuum quantities are further reinforced by the fact that, in the double-scaling limit proposed in [2], $\hat{t} = -\sum_{n\geqslant 0} \hat{t}_n \mathcal{R}_{n+1}$, where the Gelfand–Dikii polynomials $\mathcal{R}_n$ satisfy $\partial u/\partial t_m = D\mathcal{R}_{m+1}$ and are Hamiltonian densities in that $\int \mathrm{d}x\, \mathcal{R}_{m+1}$ is the Hamiltonian for the KdV flow in the variable t_m. The discrete analogue of $\hat{t}(x)$ is χ_n, which is defined by (37). The quantities $(\partial/\partial g_m)\ln(\tau_{n+1}/\tau_n)$ are also Hamiltonian densities in that $\sum_n (\partial/\partial g_m)\ln(\tau_{n+1}/\tau_n)$ is the Hamiltonian $\mathcal{H}_m$ for the Toda hierarchy (9). Furthermore, substitution of (14) into the first of the equations (8) yields $R_n = \tau_{n+1}\tau_{n-1}/\tau_n^2$, so that

$$\frac{\partial \ln R_n}{\partial g_m} = \frac{\partial}{\partial g_m}\ln\frac{\tau_{n+1}}{\tau_n} - \frac{\partial}{\partial g_m}\ln\frac{\tau_n}{\tau_{n-1}}$$

a discrete analogue of the KdV equation $\partial u/\partial t_m = D\mathcal{R}_{m+1}$.

Although fairly crude, the above analysis is indicative of the hypothesis that the links established in this paper between the supersymmetric eigenvalue model and the integrable fermionic hierarchy obtained by first variation of the Toda hierarchy are responsible for the integrable structure underlying the double-scaling limit found in [2–4].

6. Conclusion

One of the most fascinating aspects of the matrix-model approach to two-dimensional quantum gravity is that not only is the partition function a tau function for an integrable system, but it also obeys the set of Virasoro constraints $L_n Z_B[g_k; N] = 0$ ($n \geqslant -1$), with L_n defined in (4). At the level of the corresponding eigenvalue model (1), the action of the Virasoro generators L_n generates a simultaneous reparametrization of all the eigenvalues λ_i equivalent to that induced by the Virasoro operator $-\sum_{i=1}^{N} \lambda_i^{n+1} \partial/\partial\lambda_i$. Although this has no obvious analogue in the continuum limit, the action of the Virasoro generators L_n can be related to reparametrizations in another parameter space which *do* have a continuum analogue. The parameter space involved is the spectral parameter space with local coordinate λ in terms of which the orthogonal polynomials (5) are expressed. As already mentioned, the orthogonal polynomials are essentially Baker functions for the Toda hierarchy [8, 10]. In the continuum limit, the right-hand side of the recursion relation (6) satisfied by the orthogonal polynomials becomes a second-order differential operator [15] and (6) becomes the spectral equation satisfied by the continuum Baker function. It has been noted by several authors [9, 16, 17] that the Virasoro constraints on the partition function for the double-scaling limit of the one-matrix model are quite natural when viewed in terms of reparametrizations of the spectral parameter, which assumes the role of a local coordinate on a Riemann surface in the formalism of Jimbo *et al* [18] which relates tau functions for the KP hierarchy (and its two-reduction, the KdV hierarchy) to free fermion correlation functions†. The corresponding discrete analogue is the correspondence between the action of the generators L_n on the partition function of the one-matrix model and the action of the operator $\mathcal{L}_n = -\lambda^{n+1}\partial/\partial\lambda$ on the space of orthogonal polynomials [8].

For the supersymmetric eigenvalue model, the action of the super-Virasoro generators $G_{n-\frac{1}{2}}$ defined in (15) on the partition function (16) is equivalent to simultaneous superconformal transformations in the superspaces whose local coordinates are the eigenvalues (λ_i, θ_i). It is natural to enquire whether there is also some superspace, analogous to the spectral space in the bosonic case, on which Baker functions are defined and on which the super-Virasoro generators $G_{n-\frac{1}{2}}$ induce superconformal transformations. This action might then be expected to carry over to the double-scaling limit, unlike the superconformal transformations on the eigenvalues.

An obvious place to look for such a structure is on a superspace with coordinates (λ, θ), where λ and θ appear in the supersymmetric potential (17). In the case of the bosonic eigenvalue model, the Virasoro operators $\mathcal{L}_n = -\lambda^{n+1}\partial/\partial\lambda$, $n \geqslant -1$, have a natural action on the space of orthogonal polynomials because they do not involve negative powers of λ and so preserve the space of orthogonal polynomials. Are there analogues of the orthogonal polynomials in the supersymmetric case for which the action of the super-Virasoro operators $G_{n-\frac{1}{2}}$ in (15) can be linked to the action of the operators $\mathcal{G}_{n-\frac{1}{2}} = -\lambda^n[\partial/\partial\theta - \theta(\partial/\partial\lambda)]$ on the (λ, θ) superspace? Since the vertex operator construction (13) of the orthogonal polynomials is responsible for the link between the action of the Virasoro operators L_n on the partition function for the matrix model and the action of the Virasoro operators $\mathcal{L}_n$ on the orthogonal polynomials, one can attempt to construct supersymmetric analogues of the orthogonal polynomials by the action of a super-vertex operator on the partition function for the supersymmetric eigenvalue model. In fact, the objects

$$\mathcal{P}_n(\lambda, \theta) = \lambda^{2n} \frac{X(\lambda, \theta) \cdot Z_S[g_k, \xi_{k+\frac{1}{2}}; 2n]}{Z_S[g_k, \xi_{k+\frac{1}{2}}; 2n]}$$

† In fact, more general transformations of the spectral parameter generated by the action of operators of the form $\lambda^n(\partial^m/\partial\lambda^m)$ can be related to w_∞ constraints on the partition function [7, 19].

where $X(\lambda, \theta)$ is the super-vertex operator

$$X(\lambda,\theta) = \exp\left(\sum_{k>0}\frac{1}{k}\lambda^{-k}\frac{\partial}{\partial g_k} + \sum_{k\geqslant 0}\theta\lambda^{-k-1}\frac{\partial}{\partial \xi_{k+\frac{1}{2}}}\right)$$

are polynomial in positive powers of λ and θ. This is a consequence of the fact that $X(\lambda,\theta)\cdot H_i = H_i - (H_{i+1}/\lambda)$ and $X(\lambda,\theta)\cdot\zeta_{i+1} = \zeta_{i+1} - (\zeta_{i+2}/\lambda) + (\theta H_i/\lambda)$ and expression (18) for Z_S. However, as $\mathcal{P}_n(\lambda,\theta)$ has leading power λ^{2n}, the space of such polynomials is not preserved by the action of the super-Virasoro generators $\mathcal{G}_{n-\frac{1}{2}}$, $n \geqslant 0$, and the space also lacks the completeness properties of the orthogonal polynomials. Furthermore, there is no obvious relation of these polynomials in λ and θ to a Lax formulation of the integrable hierarchy (29), presumably due to the fact that, although the partition function $Z_S[g_k, \xi_{k+\frac{1}{2}}; 2N]$ is related to the hierarchy in the manner indicated earlier, it is *not* a tau function for the hierarchy.

The action of the Virasoro generators $\mathcal{L}_n = -\lambda^{n+1}\partial/\partial\lambda$ on the Baker function is very important in the analysis of the continuum limit of the matrix model, in particular in the characterization of the partition function in terms of (the square of) a solution of the Virasoro constraints [6, 7, 19, 20]. Also, it has been suggested that there is a relationship between the 'spectral' Riemann surface of the matrix models and the worldsheet of two-dimensional gravity (see [8] and section 8 of [21] for some speculations in this direction). If this is indeed the case, it is an important issue to try to identify a natural 'spectral' super-Riemann surface for the supersymmetric eigenvalue model.

Appendix A

In this appendix, it is verified that the Toda hierarchy and its first variation (the first few equations of which are given by (26)) can be expressed in the Hamiltonian form (29).

We begin by showing that the flows $\partial R_n/\partial g_m = \{\mathcal{H}_{m+1}, R_n\}_{\mathrm{PB}}$ of the ordinary Toda hierarchy can be expressed with respect to the odd Poisson bracket in the form $\partial R_n/\partial g_m = \{\hat{\mathcal{H}}_{m+1}, R_n\}_{\mathrm{AB}}$. Now

$$\{\hat{\mathcal{H}}_{m+1}, R_n\}_{\mathrm{AB}} = \sum_k \left(\frac{\partial\hat{\mathcal{H}}_{m+1}}{\partial R_k}\{R_k, R_n\}_{\mathrm{AB}} + \frac{\partial\hat{\mathcal{H}}_{m+1}}{\partial S_k}\{S_k, R_n\}_{\mathrm{AB}} + \frac{\partial\hat{\mathcal{H}}_{m+1}}{\partial \hat{R}_k}\{\hat{R}_k, R_n\}_{\mathrm{AB}} + \frac{\partial\hat{\mathcal{H}}_{m+1}}{\partial \hat{S}_k}\{\hat{S}_k, R_n\}_{\mathrm{AB}}\right). \tag{38}$$

Since

$$\hat{\mathcal{H}}_{m+1}\delta\mathcal{H}_{m+1} = \sum_k\left(\frac{\partial\mathcal{H}_{m+1}}{\partial R_k}\hat{R}_k + \frac{\partial\mathcal{H}_{m+1}}{\partial S_k}\hat{S}_k\right) \tag{39}$$

(where δ denotes an *arbitrary* first variation and $\hat{R}_n = \delta R_n$, $\hat{S}_n = \delta S_n$), it follows that

$$\frac{\partial\hat{\mathcal{H}}_{m+1}}{\partial\hat{R}_k} = \frac{\partial\mathcal{H}_{m+1}}{\partial R_k} \qquad \frac{\partial\hat{\mathcal{H}}_{m+1}}{\partial\hat{S}_k} = \frac{\partial\mathcal{H}_{m+1}}{\partial S_k}. \tag{40}$$

Using this together with $\{\hat{R}_k, R_n\}_{\rm AB} = \{R_k, R_n\}_{\rm PB}$ and a similar result with R replaced by S in (38), we obtain $\{\hat{\mathcal{H}}_{m+1}, R_n\}_{\rm AB} = \{\mathcal{H}_{m+1}, R_n\}_{\rm PB}$, which is the desired result. The same procedure can be used for the flows $\partial S_n/\partial g_m$.

To prove that $\partial \hat{R}_n/\partial g_m$ can be expressed in the Hamiltonian form (29), we begin with the expression (38) with R_n replaced by $\hat{R}_n$. Using the explicit form (30) of the odd Poisson brackets and the result (40) yields

$$\{\hat{\mathcal{H}}_{m+1}, \hat{R}_n\}_{\rm AB} = \frac{\partial \hat{\mathcal{H}}_{m+1}}{\partial S_{n-1}} R_n - \frac{\partial \hat{\mathcal{H}}_{m+1}}{\partial S_n} R_n + \frac{\partial \mathcal{H}_{m+1}}{\partial S_{n-1}} \hat{R}_n - \frac{\partial \mathcal{H}_{m+1}}{\partial S_n} \hat{R}_n. \tag{41}$$

On the other hand, variation of

$$\frac{\partial R_n}{\partial g_m} = \{\mathcal{H}_{m+1}, R_n\}_{\rm PB} = \frac{\partial \mathcal{H}_{m+1}}{\partial S_{n-1}} R_n - \frac{\partial \mathcal{H}_{m+1}}{\partial S_n} R_n$$

yields the property that $\partial \hat{R}_n/\partial g_m$ is equal to the right-hand side of (41) if the result

$$\delta \frac{\partial \mathcal{H}_{m+1}}{\partial S_n} = \frac{\partial \hat{\mathcal{H}}_{m+1}}{\partial S_n}$$

is used. This latter result follows by acting on (39) with $\partial/\partial S_n$. Similar arguments establish the expressions (29) for $\partial \hat{S}_n/\partial g_m$.

To show that the flows $\partial/\partial g_m$ and $\partial/\partial g_p$ commute, we need the Jacobi identity obeyed by the odd Poisson bracket [14]

$$0 = (-1)^{(\epsilon_P+1)(\epsilon_R+1)}\{P, \{Q, R\}_{\rm AB}\}_{\rm AB} + \text{cyclic permutations} \tag{42}$$

where $\epsilon_P = 0$ or 1 for P Grassmann even or odd, respectively. Using (29) and applying the Jacobi identity, then, for example,

$$\left[\frac{\partial}{\partial g_m}, \frac{\partial}{\partial g_p}\right] R_n = -\{R_n, \{\hat{\mathcal{H}}_{m+1}, \hat{\mathcal{H}}_{p+1}\}_{\rm AB}\}_{\rm AB}.$$

Applying (29) again, $\{\hat{\mathcal{H}}_{m+1}, \hat{\mathcal{H}}_{p+1}\}_{\rm AB} = \partial \hat{\mathcal{H}}_{p+1}/\partial g_m$, so it suffices to show that the $\hat{\mathcal{H}}_{p+1}$ are conserved by all the flows. This follows by varying the conservation law $\partial \mathcal{H}_{p+1}/\partial t_m = 0$ and using $[\delta, \partial/\partial g_m] = 0$ (the latter is true since, for example, the flow equations for $\hat{R}_n$ are obtained by varying $\partial R_n/\partial g_m$ (expressed as a polynomial in the R_i and S_i) and setting it equal to $(\partial/\partial g_m)\delta R_n$).

Next, we establish the bi-Hamiltonian form (34) of the 'odd'-flow equations (32) and (33). The results for equations (32) follow trivially from the fact that odd Poisson brackets of unhatted quantities vanish. The bi-Hamiltonian form of the equations (33) is a consequence of the result $\{\mathcal{H}_{m+1}, \hat{R}_n\}_{\rm AB} = \{\mathcal{H}_{m+1}, R_n\}_{\rm PB}$ which follows from

$$\{\mathcal{H}_{m+1}, \hat{R}_n\}_{\rm AB} = \sum_k \left(\frac{\partial \mathcal{H}_{m+1}}{\partial R_k}\{R_k, \hat{R}_n\}_{\rm AB} + \frac{\partial \mathcal{H}_{m+1}}{\partial S_k}\{S_k, \hat{R}_n\}_{\rm AB}\right)$$

and

$$\{R_k, \hat{R}_n\}_{\rm AB} = \{R_k, R_n\}_{\rm PB} \qquad \{S_k, \hat{R}_n\}_{\rm AB} = \{S_k, R_n\}_{\rm PB}.$$

Similar arguments establish the results for $\hat{S}_n$ and for the second odd Poisson bracket.

To show that the odd flows (34) anticommute with each other, application of the Jacobi identity gives, for example,

$$\frac{\partial^2 R_n}{\partial \xi_{k+\frac{1}{2}} \xi_{m+\frac{1}{2}}} + \frac{\partial^2 R_n}{\partial \xi_{m+\frac{1}{2}} \xi_{k+\frac{1}{2}}} = -\{R_n, \{\mathcal{H}_{k+1}, \mathcal{H}_{m+1}\}_{\mathrm{AB}}\}_{\mathrm{AB}}.$$

Using (34), $\{\mathcal{H}_{k+1}, \mathcal{H}_{m+1}\}_{\mathrm{AB}}$ vanishes as $\mathcal{H}_{m+1}$ has no $\xi_{k+\frac{1}{2}}$ dependence. Similar arguments apply for the flows for S_n, $\hat{R}_n$ and $\hat{S}_n$. To show that the odd flows (34) commute with the even flows (29), the Jacobi identity yields, for example,

$$\left[\frac{\partial}{\partial \xi_{k+\frac{1}{2}}}, \frac{\partial}{\partial g_m}\right] R_n = -\{R_n, \{\mathcal{H}_{k+1}, \hat{\mathcal{H}}_{m+1}\}_{\mathrm{AB}}\}_{\mathrm{AB}}.$$

The expression $\{\mathcal{H}_{k+1}, \hat{\mathcal{H}}_{m+1}\}_{\mathrm{AB}}$ vanishes, as it is equal to $-\partial\mathcal{H}_{k+1}/\partial g_m$ and the Hamiltonians $\mathcal{H}_{k+1}$ are conserved quantities for the Toda flows.

Appendix B

In this appendix, the equivalence of expressions (35) and (36) is established. Using the expression (21) for the partition function of the supersymmetric eigenvalue model and the fact that $Z_{\mathrm{B}}[g_k; N]\tau_N[g_k]$, a tau function for the Toda hierarchy, equation (35) is equivalent to

$$2\ln\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} - 2\sum_{k,m\geqslant 0} \xi_{k+\frac{1}{2}}\xi_{m+\frac{1}{2}} \frac{\partial^2}{\partial g_{k+1}\partial g_m} \ln\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}. \tag{43}$$

Substitution of (14) into the first equation of (8) leads to the expression $R_n = \tau_{n+1}\tau_{n-1}/\tau_n^2$ which allows (43) to be expressed in terms of R_n. The desired result (36) then follows by the use of the relation

$$\begin{aligned}\frac{\partial}{\partial g_{k+1}} \ln R_n = &-\frac{\partial^2}{\partial g_1 \partial g_k} \ln\frac{\tau_{n+1}}{\tau_{n-1}} \\ &+ \left(\frac{\partial}{\partial g_1}\ln\frac{\tau_n}{\tau_{n-1}}\right)\left(\frac{\partial}{\partial g_k}\ln\frac{\tau_n}{\tau_{n-1}}\right) - \left(\frac{\partial}{\partial g_1}\ln\frac{\tau_{n+1}}{\tau_n}\right)\left(\frac{\partial}{\partial g_k}\ln\frac{\tau_{n+1}}{\tau_n}\right).\end{aligned} \tag{44}$$

To establish (44), we introduce the normalized orthogonal polynomials $P_n(\lambda)$ which, by (5), are related to the unnormalized orthogonal polynomials by

$$P_n(\lambda) = \frac{p_n(\lambda)}{\sqrt{h_n}}.$$

Using $R_n = h_n/h_{n-1}$ (see, for example, [5]) and the recursion relation (6), the normalized orthogonal polynomials satisfy the recursion relation

$$\lambda P_n(\lambda) = \sqrt{R_{n+1}}P_{n+1}(\lambda) + S_n P_n(\lambda) + \sqrt{R_n}P_{n-1}(\lambda). \tag{45}$$

Also needed are the results

$$\int d\lambda\, e^{-V(\lambda)} P_n(\lambda)\lambda^k P_n(\lambda) = -\frac{\partial}{\partial g_k} \ln \frac{\tau_{n+1}}{\tau_n} \qquad (k > 0) \tag{46}$$

and

$$\int d\lambda\, e^{-V(\lambda)} P_{n-1}(\lambda)\lambda^k P_n(\lambda) = \frac{1}{\sqrt{R_n}} \frac{\partial^2}{\partial g_1 \partial g_k} \ln \tau_n \qquad (k > 0) \tag{47}$$

to be proved below. Then, equation (44) follows by applying the recursion relation (45) and the results (14), (46) and (47) to the expression

$$\int d\lambda\, e^{-V(\lambda)} (P_n(\lambda)\lambda^{k+1} P_n(\lambda) - P_{n-1}(\lambda)\lambda^{k+1} P_{n-1}(\lambda)).$$

It remains to prove (46) and (47). By differentiation of the relation

$$1 = \int d\lambda\, e^{-V(\lambda)} P_n(\lambda) P_n(\lambda)$$

with respect to g_k

$$\int d\lambda\, e^{-V(\lambda)} P_n(\lambda)\lambda^k P_n(\lambda) = 2 \int d\lambda\, e^{-V(\lambda)} P_n(\lambda) \frac{\partial P_n(\lambda)}{\partial g_k}.$$

The right-hand side of this expression is twice the coefficient of $P_n(\lambda)$ in $\partial P_n(\lambda)/\partial g_k$. Differentiating $P_n(\lambda) = (\lambda^n/\sqrt{h_n}) + O(\lambda^{n-1})$ with respect to g_k and using the standard result $h_n = \tau_{n+1}/\tau_n$ from the theory of orthogonal polynomials [5] leads to $\partial P_n(\lambda)/\partial g_k = -\frac{1}{2}(\partial/\partial g_k) \ln(\tau_{n+1}/\tau_n)$, from which the desired result follows.

To prove (47), we use the relation

$$\int d\lambda\, e^{-V(\lambda)} P_{n-1}(\lambda)\lambda^k P_n(\lambda) = (h_n h_{n-1})^{-1/2} \int d\lambda\, e^{-V(\lambda)} p_{n-1}(\lambda)\lambda^k p_n(\lambda).$$

By differentiating $0 = \int d\lambda\, e^{-V(\lambda)} p_{n-1}(\lambda) p_n(\lambda)$ with respect to g_k, the right-hand side of this expression is seen to be equivalent to $(h_n h_{n-1})^{-1/2} \int d\lambda\, e^{-V(\lambda)} p_{n-1}(\lambda) \partial p_n(\lambda)/\partial g_k$, which is $\sqrt{(h_{n-1}/h_n)} = 1/\sqrt{R_n}$ times the coefficient of $p_{n-1}(\lambda)$ in $\partial p_n(\lambda)/\partial g_k$. Using the expression (13) for $p_n(\lambda)$,

$$\begin{aligned}\frac{\partial p_n(\lambda)}{\partial g_k} &= \lambda^{n-1} \frac{\partial^2}{\partial g_1 \partial g_k} \ln \tau_n + O(\lambda^{n-2}) \\ &= p_{n-1}(\lambda) \frac{\partial^2}{\partial g_1 \partial g_k} \ln \tau_n + O(p_{n-2}(\lambda))\end{aligned}$$

from which the desired result follows.

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